QUASITORIC MANIFOLDS HOMEOMORPHIC TO HOMOGENEOUS SPACES

MICHAEL WIEMELER

Abstract. We present some classification results for quasitoric manifolds M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which admit an action of a compact connected Lie-group G such that $\dim M/G \leq 1$. In contrast to Kuroki's work [7, 6] we do not require that the action of G extends the torus action on M.

1. Introduction

Quasitoric manifolds are certain $2n$-dimensional manifolds on which an n-dimensional torus acts such that the orbit space of this action may be identified with a simple convex polytope. They were first introduced by Davis and Januszkiewicz [2] in 1991.

In [7, 6] Kuroki studied quasitoric manifolds M which admit an extension of the torus action to an action of some compact connected Lie-group G such that $\dim M/G \leq 1$. Here we drop the condition that the G-action extends the torus action in the case where the first Pontrjagin-class of M is equal to the negative of a sum of squares of elements of $H^2(M)$. In this note all cohomology groups are taken with coefficients in $\mathbb{Q}$. We have the following two results.

Theorem 1.1. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ which is homeomorphic (or diffeomorphic) to a homogeneous space G/H with G a compact connected Lie-group. Then M is homeomorphic (diffeomorphic) to $\prod S^2$. In particular, all Pontrjagin-classes of M vanish.*

Theorem 1.2. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. Assume that the compact connected Lie-group G acts smoothly and almost effectively on M such that $\dim M/G = 1$. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2-bundle over a product of two-spheres.*

The proofs of these theorems are based on Hauschild's study [4] of spaces of q-type. A space of q-type is defined to be a topological space X satisfying the following cohomological properties:

- The cohomology ring $H^*(X)$ is generated as a $\mathbb{Q}$-algebra by elements of degree two, i.e. $H^*(X) = \mathbb{Q}[x_1, \dots, x_n]/I_0$ and $\deg x_i = 2$.
- The defining ideal I_0 contains a definite quadratic form Q.

The note is organised as follows. In section 2 we show that a quasitoric manifold M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$ is of q-type. In section 3 we prove Theorem 1.1. In section 4 we recall some properties of cohomogeneity one manifolds. In section 5 we prove Theorem 1.2.

2000 *Mathematics Subject Classification.* Primary 57S15, 57S25.

Key words and phrases. quasitoric manifolds, homogeneous spaces, cohomogeneity one manifolds.

Part of the research was supported by SNF Grants Nos. 200021-117701 and 200020-126795.

The results presented in this note form part of the outcome of my Ph.D. thesis [10] written under the supervision of Prof. Anand Dessai at the University of Fribourg. I would like to thank Anand Dessai for helpful discussions.

2. Quasitoric manifolds with $p_1(M) = -\sum a_i^2$

In this section we study quasitoric manifolds M with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. To do so we first introduce some notations from [4] and [5, Chapter VII]. For a topological space X we define the topological degree of symmetry of X as

$$N_t(X) = \max\{\dim G; G \text{ compact Lie-group, } G \text{ acts effectively on } X\}$$

Similarly one defines the semi-simple degree of symmetry of X as

$$N_t^{ss}(X) = \max\{\dim G; G \text{ compact semi-simple Lie-group, } G \text{ acts effectively on } X\}$$

and the torus-degree of symmetry as

$$T_t(X) = \max\{\dim T; T \text{ torus, } T \text{ acts effectively on } X\}.$$

In the above definitions we assume that all groups act continuously.

Another important invariant of a topological space X used in [4] is the so called embedding dimension of its rational cohomology ring. For a local $\mathbb{Q}$-algebra A, we denote by $\operatorname{edim} A$ the embedding dimension of A. By definition, we have $\operatorname{edim} A = \dim_{\mathbb{Q}} \mathfrak{m}_A/\mathfrak{m}_A^2$, where $\mathfrak{m}_A$ is the maximal ideal of A. In case that $A = \bigoplus_{i\geq 0} A^i$ is a positively graded local $\mathbb{Q}$-algebra, $\mathfrak{m}_A$ is the augmentation ideal $A_+ = \bigoplus_{i>0} A^i$. If furthermore A is generated by its degree two part, then $\mathfrak{m}_A^2 = \bigoplus_{i>2} A^i$. Therefore for a quasitoric manifold M over the polytope P we have $\operatorname{edim} H^*(M) = \dim_{\mathbb{Q}} H^2(M) = m - n$ where m is the number of facets of P and n is its dimension.

Lemma 2.1. *Let M be a quasitoric manifold with $p_1(M) = -\sum a_i^2$ for some $a_i \in H^2(M)$. Then M is a manifold of q-type.*

Proof. The discussion at the beginning of section 3 of [8] together with Corollary 6.8 of [2, p. 448] shows that there are a basis $u_{n+1}, \dots, u_m$ of $H^2(M)$ and $\lambda_{i,j} \in \mathbb{Z}$ such that

$$p_1(M) = \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2.$$

Therefore

$$\begin{aligned} 0 &= \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2 + \sum_i a_i^2 \\ &= \sum_{i=n+1}^m u_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} u_i \right)^2 + \sum_j \left(\sum_{i=n+1}^m \mu_{i,j} u_i \right)^2 \end{aligned}$$

with some $\mu_{i,j} \in \mathbb{Q}$ follows.

Because

$$\sum_{i=n+1}^m X_i^2 + \sum_{j=1}^n \left(\sum_{i=n+1}^m \lambda_{i,j} X_i \right)^2 + \sum_j \left(\sum_{i=n+1}^m \mu_{i,j} X_i \right)^2$$

is a positive definite bilinear form the statement follows. □

Proposition 2.2. *Let M be a quasitoric manifold of q-type over the n-dimensional polytope P. Then we have for the number m of facets of P:*

$$m \geq 2n$$

Proof. By Theorem 3.2 of [4, p. 563], we have

$$n \leq T_t(M) \leq \operatorname{edim} H^*(M) = m - n.$$

Therefore we have $2n \leq m$. □

Remark 2.3. The inequality in the above proposition is sharp, because for $M = S^2 \times \cdots \times S^2$ we have $m = 2n$ and $p_1(M) = 0$.

By Theorem 5.13 of [4, p. 573], we have for a manifold M of q-type that $N_t^{ss} \leq \dim M + \operatorname{edim} M$. Hence, for a quasitoric manifold M, we get:

Proposition 2.4. *Let M as in Proposition 2.2. Then we have*

$$N_t^{ss}(M) \leq 2n + m - n = n + m.$$

Remark 2.5. The inequality in the above proposition is sharp because for $M = S^2 \times \cdots \times S^2$ we have $m = 2n$ and $SU(2) \times \cdots \times SU(2)$ acts on M and has dimension $3n$.

3. Quasitoric manifolds which are also homogeneous spaces

In this section we prove Theorem 1.1. Recall from Lemma 2.1 that a quasitoric manifold M with first Pontrjagin-class equal to the negative of the sum of squares of elements of $H^2(M)$ is a manifold of q-type.

Let M be a quasitoric manifold over the polytope P which is also a homogeneous space and is of q-type.

Let G be a compact connected Lie-group and $H \subset G$ a closed subgroup such that M is homeomorphic or diffeomorphic to G/H. Because $\chi(M) > 0$ and M is simply connected, we have $\operatorname{rank} G = \operatorname{rank} H$ and H is connected. Therefore we may assume that G is semi-simple and simply connected.

Let T be a maximal torus of G. Then $(G/H)^T$ is non-empty. By Theorem 5.9 of [4, p. 572], the isotropy group G_x of a point $x \in (G/H)^T$ is a maximal torus of G. Hence, H is a maximal torus of G.

Now it follows from Theorem 3.3 of [4, p. 563] that

$$T_t(G/H) = \operatorname{rank} G.$$

Because M is quasitoric, we have $n \leq T_t(G/H)$. Combining these inequations, we get

$$\dim G - \dim H = \dim M = 2n \leq 2 \operatorname{rank} G.$$

This equation implies that $\dim G \leq 3 \operatorname{rank} G$.

For a simple simply connected Lie-group G' we have $\dim G' \geq 3 \operatorname{rank} G'$ and $\dim G' = 3 \operatorname{rank} G'$ if and only if $G' = SU(2)$. Therefore we have $G = \prod SU(2)$ and $M = \prod SU(2)/T^1 = \prod S^2$. This proves Theorem 1.1.

4. Cohomogeneity one manifolds

Here we discuss some facts about closed cohomogeneity one Riemannian G-manifolds M with orbit space a compact interval $[-1, 1]$. We follow [3, p. 39-44] in this discussion.

We fix a normal geodesic $c : [-1, 1] \to M$ perpendicular to all orbits. We denote by H the principal isotropy group $G_{c(0)}$, which is equal to the isotropy group $G_{c(t)}$ for $t \in]-1, 1[$, and by $K^{\pm}$ the isotropy groups of $c(\pm 1)$.

Then M is the union of tabular neighbourhoods of the non-principal orbits $Gc(\pm 1)$ glued along their boundary, i.e., by the slice theorem we have

$$M = G \times_{K^-} D_- \cup G \times_{K^+} D_+, \tag{4.1}$$

where $D_{\pm}$ are discs. Furthermore $K^{\pm}/H = \partial D_{\pm} = S_{\pm}$ are spheres.

Note that M may be reconstructed from the following diagram of groups.

$$\begin{tikzcd}
& G & \\
K^- \arrow[ur] & & K^+ \arrow[ul] \\
& H \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The construction of such a group diagram from a cohologeneity one manifold may be reversed. Namely, if such a group diagram with $K^\pm/H = S_\pm$ spheres is given, then one may construct a cohomogeneity one G-manifold from it. We also write these diagrams as $H \subset K^-, K^+ \subset G$.

Now we give a criterion for two group diagrams yielding up to G-equivariant diffeomorphism the same manifold M.

Lemma 4.1 ([3, p. 44])**.** *The group diagrams $H \subset K^-, K_1^+ \subset G$ and $H \subset K^-, K_2^+ \subset G$ yield the same cohomogeneity one manifold up to equivariant diffeomorphism if there is an $a \in N_G(H)^0$ with $K_1^+ = aK_2^+a^{-1}$.*

5. Quasitoric manifolds with cohomogeneity one actions

In this section we study quasitoric manifolds M which admit a smooth action of a compact connected Lie-group G which has an orbit of codimension one. As before we do not assume that the G-action on M extends the torus action. We have the following lemma:

Lemma 5.1. *Let M be a quasitoric manifold of dimension $2n$ which is of q-type. Assume that the compact connected Lie-group G acts almost effectively and smoothly on M such that $\dim M/G = 1$. Then we have:*

1. *The singular orbits are given by G/T where T is a maximal torus of G.*
2. *The Euler-characteristic of M is $2\#W(G)$.*
3. *The principal orbit type is given by G/S, where $S \subset T$ is a subgroup of codimension one.*
4. *The center Z of G has dimension at most one.*
5. $\dim G/T = 2n - 2$.

Proof. At first note that M/G is an interval $[-1, 1]$ and not a circle because M is simply connected. We start with proving (1). Let T be a maximal torus of G. By passing to a finite covering group of G we may assume $G = G' \times Z'$ with G' a compact connected semi-simple Lie-group and Z' a torus. Let $x \in M^T$. Then the isotropy group G_x has maximal rank in G. Therefore G_x splits as $G'_x \times Z'$.

By Theorem 5.9 of [4, p. 572], G'_x is a maximal torus of G'. Therefore we have $G_x = T$.

Because $\dim G - \dim T$ is even, x is contained in a singular orbit. In particular we have

$$\chi(M) = \chi(M^T) = \chi(G/K^+) + \chi(G/K^-), \tag{5.1}$$

where $G/K^\pm$ are the singular orbits. Furthermore we may assume that G/K^+ contains a T-fixed point. This implies

$$\chi(G/K^+) = \chi(G/T) = \#W(G) = \#W(G'). \tag{5.2}$$

Now assume that all T-fixed points are contained in the singular orbit G/K^+. Then we have $(G/K^-)^T = \emptyset$. This implies

$$\chi(M) = \chi(G/K^+) = \#W(G').$$

Now Theorem 5.11 of [4, p. 573] implies that M is the homogeneous space $G'/G'\cap T = G/T$. This contradicts our assumption that $\dim M/G = 1$.

Therefore both singular orbits contain T-fixed points. This implies that they are of type G/T. This proves (1). (2) follows from (5.1) and (5.2).

Now we prove (3) and (5). Let $S \subset T$ be a minimal isotropy group. Then T/S is a sphere of dimension $\operatorname{codim}(G/T, M) - 1$. Therefore S is a subgroup of codimension one in T and $\operatorname{codim}(G/T, M) = 2$.

If the center of G has dimension greater than one, then $\dim Z' \cap S \geq 1$. That means that the action is not almost effective. Therefore (4) holds. □

By Lemma 5.1, we have with the notation of the previous section that $K^{\pm}$ are maximal tori of G containing $H = S$. In the following we will write $G = G' \times Z'$ with G' a compact connected semi-simple Lie-group and Z' a torus.

Because $K^{\pm}$ are maximal tori of the identity component $Z_G(S)^0$ of the centraliser of S, there is some $a \in Z_G(S)^0$ such that $K^- = aK^+a^{-1}$. By Lemma 4.1, we may assume that $K^+ = K^- = T$. Now from Theorem 4.1 of [9, p. 198] it follows that M is a fiber bundle over G/T with fiber the cohomogeneity one manifold with group diagram $S \subset T, T \subset T$. Therefore it is a S^2-bundle over G/T.

Lemma 5.2. *Let M and G as in the previous lemma. Then we have*

$$T_t(M) \leq \operatorname{rank} G' + 1.$$

Proof. At first we recall the rational cohomology of G/T. By [1, p. 67], we have

$$H^*(G/T) \cong H^*(BT)/I$$

where I is the ideal generated by the elements of positive degree which are invariant under the action of the Weyl-group of G. Therefore it follows that

$$\dim_{\mathbb{Q}} H^{\text{odd}}(G/T) = 0 \qquad \text{and} \qquad \dim_{\mathbb{Q}} H^2(G/T) = \operatorname{rank} G'.$$

Therefore the Serre spectral sequence for the fibration $S^2 \to M \to G/T$ degenerates. Hence, we have

$$H^*(M) = H^*(G/T) \otimes H^*(S^2)$$

as $H^*(G/T)$-modules. In particular, we have

$$\dim_{\mathbb{Q}} H^2(M) = \dim_{\mathbb{Q}} H^2(G/T) + \dim_{\mathbb{Q}} H^2(S^2) = \operatorname{rank} G' + 1.$$

Therefore

$$T_t(M) \leq \operatorname{edim} H^*(M) = \dim_{\mathbb{Q}} H^2(M) = \operatorname{rank} G' + 1$$

follows. □

Theorem 5.3. *Let M and G as in the previous lemmas. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2-bundle over a product of two-spheres.*

Proof. Because M is quasitoric we have $n \leq T_t(M)$. By Lemma 5.1 we have

$$\dim G' - \operatorname{rank} G' = \dim G/T = 2n - 2.$$

Now Lemma 5.2 implies

$$\dim G' = 2n - 2 + \operatorname{rank} G' \leq 3 \operatorname{rank} G'.$$

Therefore $\prod SU(2)$ is a finite covering group of G'. This implies the statement about the finite covering group of G.

It follows that $G/T = \prod S^2$. Therefore M is a S^2-bundle over $\prod S^2$. □

Now Theorem 1.2 follows from Theorem 5.3 and Lemma 2.1.

References

[1] A. Borel, *Topics in the homology theory of fibre bundles.*, Berlin-Heidelberg-New York: Springer-Verlag, 1967 (English).

[2] M. Davis and T. Januszkiewicz, *Convex polytopes, coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451.

[3] K. Grove, B. Wilking, and W. Ziller, *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry.*, J. Differ. Geom. **78** (2008), no. 1, 33–111 (English).

[4] V. Hauschild, *The Euler characteristic as an obstruction to compact Lie group actions.*, Trans. Am. Math. Soc. **298** (1986), 549–578 (English).

[5] W. Y. Hsiang, *Cohomology theory of topological transformation groups.*, Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 85. Berlin-Heidelberg-New York: Springer-Verlag., 1975 (English).

[6] S. Kuroki, *Classification of quasitoric manifolds with codimension one extended actions*, Preprint (2009).

[7] ———, *Characterization of homogeneous torus manifolds*, Osaka J. Math. **47** (2010), no. 1, 285–299 (English).

[8] M. Masuda and T. E. Panov, *Semi-free circle actions, Bott towers, and quasitoric manifolds*, Mat. Sb. **199** (2008), no. 8, 95–122.

[9] J. Parker, *4-dimensional G-manifolds with 3-dimensional orbits.*, Pac. J. Math. **125** (1986), 187–204 (English).

[10] M. Wiemeler, *On the classification of torus manifolds with and without non-abelian symmetries*, Ph.D. thesis, University of Fribourg, 2010.

School of Mathematics, Alan Turing Building, The University of Manchester, Oxford Road, Manchester M13 9PL, UK
E-mail address: michael.wiemeler-2@manchester.ac.uk